Calculation of kinematic characteristics of metal forming processes in lagrangian coordinates

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The energy model of a deformation object, which is a densely packed system of material particles, provides a fundamental basis for applying the mathematical apparatus of continuous functions. This approach allows for an adequate description of the complex behavior of materials in metal forming processes. When an external force is applied by the tool to the system of particles in the workpiece, the equations of motion formulated in lagrangian coordinates play a key role. Their main advantage lies in the ability to account for the individual loading history of each fixed material particle, which is crucial for analyzing the deformation state of the object. The lagrangian equations not only determine the trajectory and specific motion of each particle but also allow for establishing its precise position in space, described by eulerian coordinates. Within the framework of the model, computational relations were proposed for determining the partial derivatives of the Eulerian coordinates with respect to the Lagrangian variables and time. Based on these, an effective methodology for calculating the kinematic invariants of the motion of material particles was developed. These invariants objectively characterize the deformation state, independent of the reference frame. The use of the Lagrangian description is methodologically preferential because fundamental physical laws are formulated precisely for material particles, not for points in space. The practical significance of the model is confirmed by obtaining invariant characteristics of the deformed state, using examples of plane and axisymmetric upsetting. It is important to note that the equations of motion in lagrangian coordinates open up the possibility of applying the powerful method of superposition. This is especially relevant for modeling complex, combined processes where it is necessary to account for the superposition of several types of deformation. This approach significantly improves the accuracy of prediction and the optimization of technological parameters.
Key words: metal forming, stress, strain, lagrangian coordinates, eulerian coordinates, upsetting, strain path, shear strain rate intensity, shear strain intensity
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Introduction

The laws of motion in mechanics are essential for structural calculations and are also used in meteorology, medicine (prosthetics), biology, and chemistry. These laws must be applied in the calculations of technological processes across all branches of metallurgy and in metal forming (OMD) [1–3]. In continuum mechanics, Eulerian and Lagrangian coordinates are used to describe the motion of particles within a medium [4–7]. Equations in Eulerian coordinates allow for determining the state of the medium at the current moment in time. They are equilibrium equations and do not account for the action of body forces or the acceleration of particles during deformation. Known methods, such as methods for the simultaneous solution of differential equations of equilibrium and plasticity, the slip line method (lower bound estimate of forces), and the upper bound method based on kinematically admissible velocity fields, fundamentally apply the axioms of statics, which do not consider time. The modern approach to solving the stress-strain state using the finite element method [8, 9] is also based on the aforementioned methods, utilizing equilibrium equations and the variational principle when formulating the governing equations. This approach does not allow for obtaining the equations of motion for fixed particles of the medium, and in some cases, additional conditions are required to obtain correct results.

The application of Lagrangian coordinates makes it possible to focus on the particle and answer the question of what happens to it over time. This method can be applied when solving problems using the finite element method. In metal forming, the use of Lagrangian coordinates allows for considering the loading history of particles, which is particularly important for solving problems related to the influence of deformation conditions on material work hardening, its plasticity, recrystallization processes, the level of structure-sensitive properties, as well as local thermal effects [10] accompanying plastic deformation. Using Lagrangian variables, the stress-strain state can be determined from experimental trajectories. To meet modern requirements for finished product quality, it is necessary to increase the reliability of experimental data concerning the invariant characteristics of the stress and strain states, obtained within the volume of products, taking into account the loading history.

Initial data and method of analysis

Axisymmetric upsetting (ASU) and plane strain upsetting (PSU) are the most common processes in metal forming, and they are also widely used in laboratory material testing [11, 12].

By applying the Lagrangian approach based on experimental particle trajectory data, the primary kinematic characteristics were determined: the shear strain rate intensity – S_e and the shear strain intensity – γ_e. This research objective was set to investigate upsetting (ASU) and extrusion (PSU) processes.

Lead was used as the model material because it satisfactorily simulates the behavior of steel during hot metal forming (e.g., in rolling, die forging, hammer forging, extrusion). Consequently, lead billets were utilized to manufacture the ASU and PSU specimens.

Concurrently, specimens made of steel 40Kh (41Cr4) and aluminum AD1 (1050A), subjected to hot deformation, were also investigated in the experiments.

When investigating the stress state, it is necessary to use hardening models that depend on temperature – T, the shear stress intensity – τ_e, and the shear strain rate intensity – S_e.

$$\tau_e = \tau_e (S_e, \tau_e, T) \quad (1)$$

This article focuses on the strain state of upsetting and hot stamping processes for 40Kh steel specimens.

To eliminate the influence of internal defects, the specimens were manufactured from extruded billets with a degree of deformation of 60–70 %. The initial height of the plane and cylindrical specimens (**Fig. 1**) was H_0 = 50 mm. The dimensions of the plane specimens were (height × width × length: 50×50×48 mm), and the cylindrical specimens were (height x diameter: 50×50 mm). The obtained specimens were upset to a height of h = 25 mm, with uniform time steps. To record the initial and current coordinates of the particles, a coordinate grid was applied to the deformation plane prior to upsetting.

The grid has a step size of $\Delta\alpha$ = 5 mm and $\Delta\beta$ = 5 mm. The grid nodes correspond to the initial coordinates of the particles during the deformation of the specimens over five loading stages: Δt = 5 s, with a constant velocity V_0 = 1 mm/s. The upsetting was carried out using rough plates, which approximately corresponds to a Coulomb friction coefficient μ = 0.5.

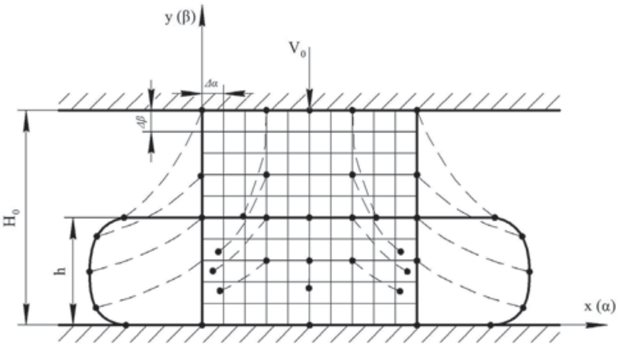

Fig. 1. Particle trajectories during upsetting

For each grid node, the measurement of the initial and current coordinates was carried out with an accuracy of up to 0.01 mm. After measuring the coordinates at each upsetting step, five points were recorded on the particle trajectories corresponding to the upsetting step. The particle trajectories were obtained by interpolation using the least squares method with a second-order polynomial (see Fig. 1).

The particle trajectories can be expressed in Lagrangian form (2) and in Eulerian form (3):

$$x_i = x_i\left(\alpha_{p,t}\right) \quad (2)$$

where $x_i \in (x, y, z)$, $\alpha_p \in (\alpha, \beta, \gamma)$, for t=0 $\alpha = x_0; \beta = y_0; \gamma = z_0$

$$\alpha_p = \alpha_p\left(x_{i,} t\right) \quad (3)$$

If the equation of motion is given in the form (3), then differentiation with respect to time dt is carried out according to the rules for differentiating implicit functions:

$$\frac{df\left(x_{i,} t\right)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\cdot v_x + \frac{\partial f}{\partial y}\cdot v_y + \frac{\partial f}{\partial z}\cdot v_z \quad (4)$$

The expression (4) is the total, material derivative of a particle, or the substantial derivative. When the motion function is specified in Lagrangian coordinates, differentiation with respect to time equals the partial derivative (5):

$$\frac{df\left(\alpha, \beta, \gamma, t\right)}{dt} = \frac{\partial f}{\partial t} \quad (5)$$

To calculate the components of particle displacement velocity in the Cartesian coordinate system in the horizontal direction of the grid u=xt and in the vertical direction of the grid v= yt, the derivative formulas (6)–(10) are presented.

The derivatives for the first points of the trajectory are determined by equation (6):

$$f' = \frac{1}{20\Delta}\cdot\left(-21 f(0) + 13 f(\Delta) + 17 f(2\Delta) - 9 f(3\Delta)\right) \quad (6)$$

The derivatives for the second points of the trajectory are calculated using formula (7).

$$f' = \frac{1}{20\Delta}\cdot\left(-11 f(-\Delta) + 3 f(0) + 7 f(\Delta) + f(2\Delta)\right) \quad (7)$$

For the penultimate points of the trajectories, the derivatives are calculated using the formula:

$$f' = \frac{1}{20\Delta}\cdot\left(11 f(\Delta) - 3 f(0) - 7 f(-\Delta) - f(-2\Delta)\right) \quad (8)$$

The calculation of derivatives for the last points of the trajectories is determined by the following formula:

$$f' = \frac{1}{20\Delta}\cdot\left(21 f(0) - 13 f(-\Delta) - 17 f(-2\Delta) + 9 f(-3\Delta)\right) \quad (9)$$

For all trajectories, the first derivatives of the central points are calculated using formula (10)

$$f' = \frac{1}{10\Delta}\cdot\left(-2 f(-2\Delta) - f(-\Delta) + f(\Delta) + 2\cdot f(2\Delta)\right) \quad (10)$$

where $f(0)$ – is the value of the grid node coordinates at the considered stage of upsetting;

$f(-\Delta); f(-2\Delta); f(-3\Delta)$ – are the values of the grid node coordinates at the previous three stages of upsetting;

$f(\Delta); f(2\Delta); f(3\Delta)$ – are the values of the grid node coordinates at the next three stages of upsetting.

Next, the derivatives of the Eulerian coordinates with respect to the Lagrangian coordinates x_α, x_β, y_α, y_β and the mixed derivatives $x_{t\alpha}$, $x_{t\beta}$, $y_{t\alpha}$, $y_{t\beta}$ were determined, according to the same formulas (6)–(10). In this case:

$f(0)$ – is the current value of the grid node coordinates at the corresponding α and β;

$f(-\Delta)$; $f(-2\Delta)$; $f(-3\Delta)$ – are the values of the coordinates of the previous grid nodes at the corresponding α and β;

$f(\Delta)$; $f(2\Delta)$; $f(3\Delta)$ – are the values of the coordinates of the subsequent grid nodes at the corresponding α and β.

Results and analysis

For technological calculations, the incompressibility condition ($R = 1$) is usually calculated. The change in the infinitesimal volume of a particle (R-1) can be expressed through the determinant of the matrix (11), the components of which are the derivatives of the Eulerian coordinates with respect to the Lagrangian coordinates [13]:

$$R = \frac{\partial V}{\partial V_0} = det\left|x_{ip}\right| = \begin{vmatrix} x_\alpha & x_\beta & x_\gamma \\ y_\alpha & y_\beta & y_\gamma \\ z_\alpha & z_\beta & z_\gamma \end{vmatrix} \quad (11)$$

The condition R-1=0 makes it possible to quantitatively estimate the error of the initial information when measuring the grid coordinates and determining the derivatives using equations (6)–(10) at each point of the particle trajectory (**Fig. 2**). In all figures, the numbers indicate the corresponding Lagrangian coordinates of the particle trajectories.

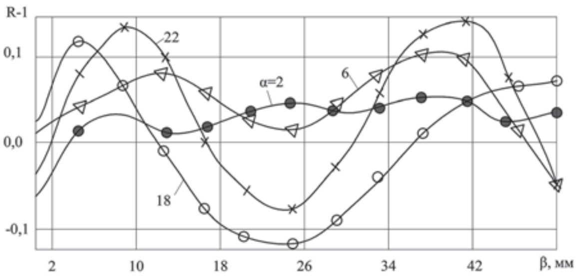

Figure 2 – Relative change in the volume of particles (R-1) during plane upsetting of lead specimens

The values of (R-1) in most cases fall within the interval of +0.1 and –0.15, which corresponds to a volume change of no more than +10 % and no less than –15 %. This allows us to note that the methods for determining trajectories for upsetting processes are quite satisfactory.

The particle displacement velocities within the workpiece volume were determined using equations (7)–(11). Graphs of the relative velocities y_t / V_0 , x_t / V_0 are presented for plane upsetting in **Fig. 3**. The course of the displacement velocity curves, by the nature of their distribution, can serve as an assessment of the logical consistency of the obtained characteristics.

The vertical displacement velocity (y_t, z_t) is characterized by a more uniform velocity distribution. Moreover, the values of y_t and z_t in the center and at the contact can serve as a guideline for the correctness of the calculations. Thus, during plane upsetting on a stationary tool surface, the ratio y_t / v_0 is equal to 0. In the center of the specimens, $y_t / v_0 = 0{,}5$, and on the movable plate, $y_t / v_0 = 1$, which is confirmed by the calculation (Fig. 3 *a*, *b*). The same is typical for axisymmetric upsetting.

The trend of the curves for x_t / v_0 and ρ_t / v_0 is similar for both processes. The minimum values of x_t and ρ_t on the stationary contact surface are 0 ($x_t = \rho_t = 0$). The curves increase monotonically when moving towards the horizontal plane of symmetry. The maximum value of the horizontal displacement velocity is achieved on the lateral surface at the point of intersection with the horizontal plane of symmetry.

Presented below are the formulas for calculating the invariants [9] that characterize plastic deformation, which include: e – average strain (12); Γ^2 – root-mean-square change in the dimensions of the edges of a parallelepiped of an infinitesimal volume of particles (13); Γ_e^2– equivalent strain (14):

$$e = \frac{e_\alpha + e_\beta + e_\gamma}{3} \quad (12)$$

$$\Gamma^2 = (e_\alpha - e)^2 + (e_\beta - e) + e_\gamma - e) \quad (13)$$

$$\Gamma_e^2 = 3e^2 + \Gamma^2 \quad (14)$$

Plastic deformation is absent if these invariants take the following values: $e_p = 1$; $R = 1$; $\Gamma_e^2 = 0$

An important characteristic for determining the quality of the finished product is the invariant quantity – the shear strain rate, S_e. **Fig. 4** (*a, b*) shows the distribution of S_e over the volume of the specimen for PSU with a strain degree of ε = 5.8 % and ε = 50 %, respectively. Studies were also conducted for ASU with strain degrees of ε = 17.4 % and ε = 33 %, respectively.

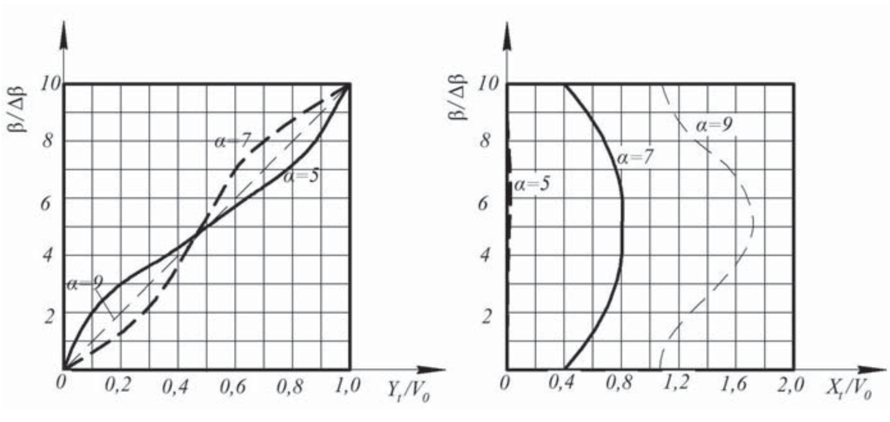

Fig. 3. Components of particle velocity for the last stage of upsetting $\Delta h/H_0$=50 %:
$a - y_t / v_0$; $b - x_t / v_0$

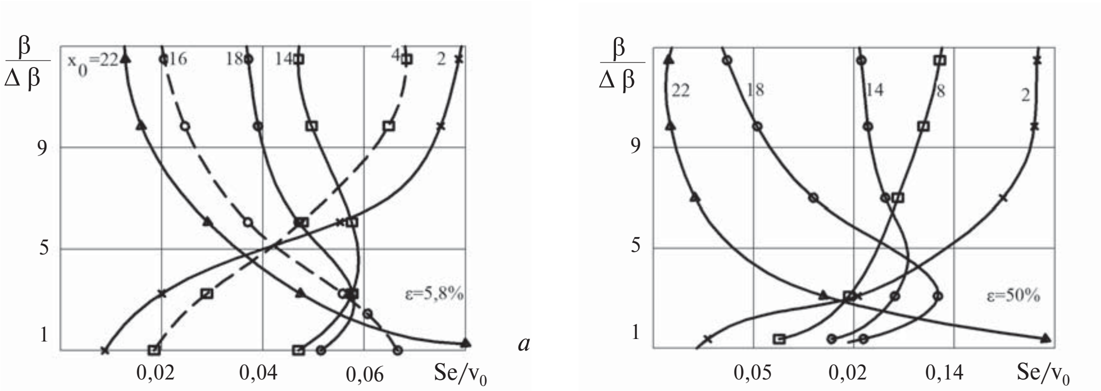

Fig. 4. Influence of the strain degree $\varepsilon = \Delta h/H_0$ on the distribution of Se during upsetting of lead specimens: *a* – PSU, ε = 5.8 %; *b* – PSU, ε = 50 %

As can be seen from the graphs in Fig. 4, the nature of the Se distribution in the volume of the specimens for PSU and ASU is the same and is practically independent of the strain degree. The shear strain intensity γ_e was obtained by integrating the increment of the strain rate intensity along the trajectory (15). The accumulated strain γ_e (Odqvist parameter) is related to the work of external forces at the contact and is presented in **Fig. 5**.

$$\gamma_e = \int S_e dt \tag{15}$$

A comparison of the strain and strain rate diagrams throughout the entire volume reveals a sharply heterogeneous pattern and confirms the possibility of modeling the deformed state for steel using lead. The heterogeneity in the distribution of strain intensity and strain rate intensity depends on contact friction. At maximum and minimum contact friction, the values of γ_e and S_e for the center of the specimens differ by a factor of 1.5 (for ASU) and by a factor of 2 (for PSU). As the reduction increases, the difference between the values of these quantities for ASU and PSU decreases. Overall, the obtained results are characteristic of PSU and ASU processes and correlate with other experimental data.

The proposed method made it possible to calculate the deformation parameters (strain intensity γ_e) within the volume of parts obtained during the forging of a gear (made of 40Kh (41Cr4) steel). One can see in **Fig. 6** the pronounced heterogeneity of the deformed state, which makes it possible to predict defects in forgings such as cracks, as well as the processes of structure formation and other product quality parameters [10].

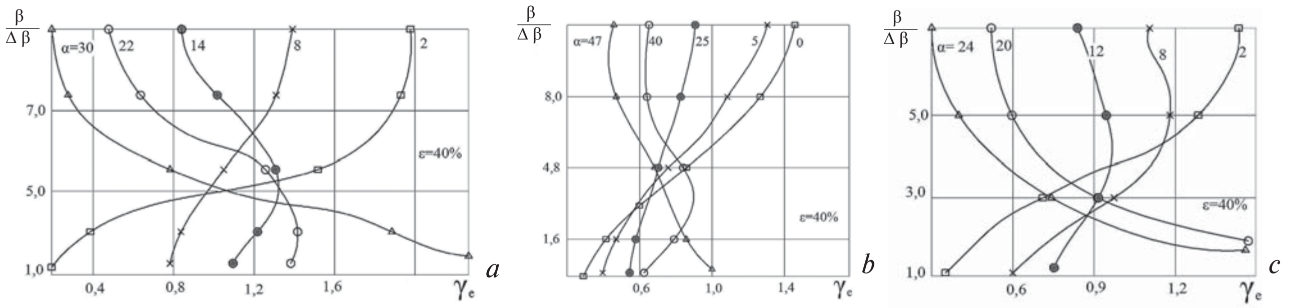

Fig. 5. Distribution of γ_e in the volume with an upsetting value of ε = 40 % for a lead specimen under ASU (*a*), a steel specimen under ASU (*b*), and an aluminum alloy specimen 40Kh (41Cr4) under PSU (*c*)

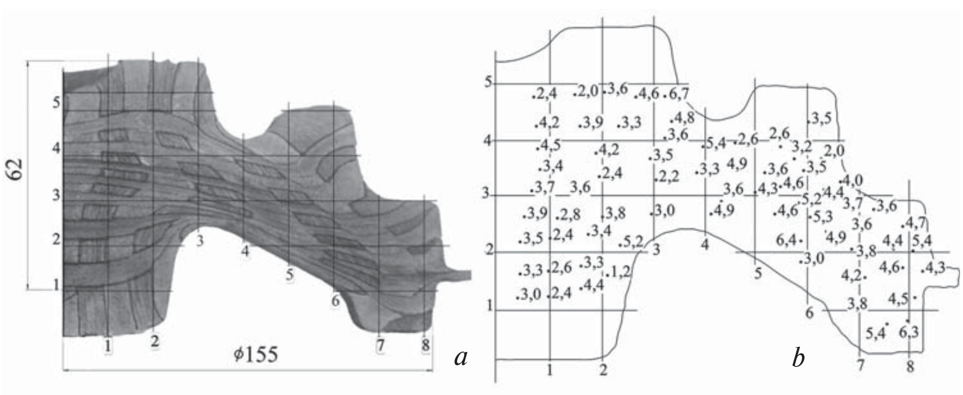

Fig. 6. Photograph of a gear forging with a diameter of 155 mm and a height of 62 mm (*a*) and the distribution of shear strain intensity γ_e within the volume of the forging (*b*)

Conclusions

Equations have been proposed for calculating the derivatives of Eulerian coordinates with respect to Lagrangian coordinates and time, which form the analytical basis for determining the complete set of kinematic invariant characteristics of any plastic deformation process. It has been established that the shear strain intensity γ_e, being an energy measure, requires consideration of the loading history, which is correctly implemented when using Lagrangian coordinates in the equations of particle motion. The application of the Lagrangian approach opens up the possibility of using the superposition method for the kinematic analysis of combined metal forming processes (e.g., screw rolling).

Based on this methodology, it becomes possible to predict the structural and mechanical characteristics of steel, including grain size heterogeneity, plasticity resource, and local thermal effects in particles.

It is also applicable for modeling combined metal forming processes that require consideration of the superposition of heterogeneous deformation fields, which significantly increases the accuracy of forecasting and the optimization of technological parameters.

The results of the conducted research can be applied in the development of technological processing modes for new materials at ferrous metallurgy enterprises. In particular, they are relevant for longitudinal and screw rolling shops, as well as for forging and stamping processes at enterprises such as JSC "VMZ," JSC "VTZ," JSC "VrTZ," JSC "ChTPZ," JSC "ITZ," JSC "ZTZ," PJSC "MMK," and others.

The developed method for calculating deformation parameters in metal forming serves as an effective alternative to expensive commercial simulation software packages (such as QForm and Deform) or extends their functionality as a subroutine. The application of this method reduces economic and time costs due to optimized algorithms, ensuring high simulation accuracy and reducing production preparation time.

REFERENCES

1. Troitskii D. V., Gamin Yu. V., Galkin S. P., Budnikov A. S. Parametric model of a three-roll unit of radial-shear rolling mini-mill. *Izvestiya. Ferrous Metallurgy*. 2023. Vol. 66 No. 3. pp. 376–386. DOI: 10.17073/0368-0797-2023-3-376-386.
2. Galkin S. P., Romantsev B. A., Kharitonov E. A. Realization of innovation potential of the universal method of radial-shift rolling. *Chernye Metally*. 2015. No. 1. pp. 23–28.
3. Shinkin V. N., Kolikov A. P. Engineering calculations for processes involved in the production of large-diameter pipes by the SMS Meer technology. *Metallurgist*. 2012. Vol. 55. No. 11–12. pp. 833–840.
4. Guilkey, B., Weiss J. A. On Lagrangian mechanics and the implicit material point method for large deformation elasto-plasticity. *Computer Methods in Applied Mechanics and Engineering*. 2020. Vol. 372. 113343. DOI: 10.1016/j.cma.2020.113343.
5. Lee J. A Skew-Symmetric Plastic Potential Model for Material Processing Lagrangian Formulation. *International Journal of Precision Engineering and Manufacturing*. 2025.
6. Freed A. On the use of Eulerian and Lagrangian Laplace stretches. Journal of Strain Analysis for Engineering Design. 2020. Vol. 55. No. 5-6. pp. 219-233. DOI: 10.1177/0309324720911995.
7. Freed A. D., Zamani S., Szabo L., Clayton J. D. Laplace Stretch: Eulerian and Lagrangian Formulations. *Zeitshrift für angewandte Mathematik und Physik*. 2020. DOI: 10.48550/arXiv.2003.06376.
8. Gamin Yu. V., Koshmin A. N., Kin T. Yu., Aleshchenko A.S. Comparative analysis of stress-strain state of bars from aluminum alloys A2024 and A7075 processed by RSR based on FEM modeling. *Materials Today: Proceedings*.2021. Vol. 46. Part 17. pp. 8138–8142. DOI: 10.1016/j.matpr.2021.03.106.
9. Shinkin V. N., Kolikov A. P. Simulation of the shaping of blanks for large-diameter pipe. *Steel in Translation*. 2011. Vol. 41. No. 1. pp. 61–66.
10. Shinkin V. N. Simple analytical dependence of elastic modulus on high temperatures for some steels and alloys. *CIS Iron and Steel Review*. 2018. Vol. 15. pp. 32–38.
11. Alyushin Y. A. Energy scale of the average stresses and physical properties of metals in the region of reversible and irreversible deformations. *Journal of Machinery Manufacture and Reliability*. 2010. Vol. 39. No. 3. pp. 282–289.
12. Alyushin Yu. A., Gorbatyuk S. M. Possible mechanisms of energy dissipation in the transition from reversible deformation to irreversible. *Izvestiya. Ferrous Metallurgy*. 2018. Vol. 61. No. 3. pp. 223–229. DOI:10.17073/0368-0797-2018-3-223-229.
13. Alyushin Y. A. Energy mechanisms of free vibrations and resonance in elastic bodies. *Physics (Switzerland)*. 2021. Vol. 3. No. 4. pp. 1133–1154.